2019

MATHEMATICS

(Major)

Paper : 4.1

(Real Analysis)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following as directed : 1×10=10

(a) Let S be a nonempty subset of R that is bounded below. Then choose the correct option for

$$k = \sup S / \inf S / -\sup S / -\inf S$$

if $k = -\sup\{-s \in S\}$.

(b) If $G_n = (0, 1+1/n)$ for $n \in N$, then the intersection $\bigcap_{n=1}^{\infty} G_n$ is open.

(Write True or False)

(c) For $\{x_n\}$ given by the formula $x_n = n/(n+1)$, establish either the convergence or the divergence of the sequence $\{x_n\}$.

(d) If β a limit point of a sequence $\{S_n\}$, then there exists a subsequence $\{S_{n_k}\}$ of $\{S_n\}$ which converges to β.

(Write True or False)

(e) If Σu_n is a positive term series such that $\lim_{n\to\infty} (u_n)^{1/n} = L$, then under what condition the Cauchy's root test confirms divergence of Σu_n?

(f) The series

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\cdots$$

is not convergent. Give reason.

(g) Define limit of a function (sequential approach).

(h) Is the function f, where $f(x) = \dfrac{x-|x|}{x}$ continuous?

(i) If the function defined on the closed interval $[a, b]$ satisfies the conditions of the mean-value theorem and $f'(x) = 0$ for all $x \in]a, b[$, then verify that $f(x)$ is constant on $[a, b]$.

(j) A function f is defined on R by

$$f(x) = \begin{cases} x, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x \ge 1 \end{cases}$$

Is $f'(1)$ exist? Justify.

2. Answer the following questions : 2×5=10

(a) Any open interval $I = (a, b)$ is an open set. Why?

(b) Show that the series $\sum \frac{1}{n}$ does not converge.

(c) Prove that if f is continuous on $[a, b]$ and $f(x) \in [a, b]$ for every $x \in [a, b]$, then there exists a point $c \in [a, b]$ such that $f(c) = c$.

(d) Show that the maximum value of $(\log x)/x$ in $0 < x < \infty$ is $1/e$.

(e) Show that the function $f(x) = x^{1/3}$, $x \in R$, is not differentiable at $x = 0$.

3. Answer any *four* parts : 5×4=20

(a) Prove that the intersection of any finite number of open sets is open. Does this result hold for arbitrary family of open sets? Justify it.

(b) If $\{a_n\}$ and $\{b_n\}$ be two sequences such that $\lim a_n = a$ and $\lim b_n = b$, then prove that $\lim(a_n b_n) = ab$.

(c) Prove that a positive term series Σu_n, where $u_n = \frac{1}{n^p}$, is convergent if $p > 1$.

(d) Test for convergence of the series

$$\sum \frac{(n!)}{(2n)!} x^n, \ x > 0$$

(e) Prove that a function f, which is continuous on a closed interval $[a, b]$, assumes every value between its bounds.

(f) Expand, if possible, the function $f(x) = \sin x$ in ascending powers of x.

4. Answer *either* (a) and (b) *or* (c) and (d) : 5×2=10

(a) State and prove Bolzano-Weierstrass theorem for sets. 1+4=5

(b) Show that the sequence $\{a_n\}$, where

$$a_n = \left\{\frac{1}{\sqrt{(n^2+1)}} + \frac{1}{\sqrt{(n^2+2)}} + \cdots + \frac{1}{\sqrt{(n^2+n)}}\right\}$$

converges to 1. 5

(c) If $\{b_n\}$ be a sequence of positive real numbers such that $b_n = \sqrt{b_{n-1} b_{n-2}}$, $n > 2$, then show that the sequence converges to $(b_1 b_2^2)^{1/3}$. 5

(d) Prove that a monotonic increasing bounded above sequence converges to its least upper bound. 5

5. Answer *either* (a) and (b) *or* (c) and (d) : 5×2=10

(a) State Abel's test and show that

$$1 - \frac{1}{3 \cdot 2^2} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 4^2} + \cdots$$

is convergent. 1+4=5

(b) Test for convergence of the following series whose nth term is given by

$$\frac{1.3.5 \cdots (4n-3)}{2.4.6 \cdots (4n-2)} \cdot \frac{x^{2n}}{4n}$$ 5

(c) State the comparison test (limit form) and using it, test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$. 1+4=5

(d) When an infinite series Σu_n is said to be absolutely convergent? Prove that if Σu_n absolutely convergent, then $\Sigma |u_n|$ is convergent. Does the divergence of $\Sigma |u_n|$ imply the divergence of Σu_n? 1+3+1=5

6. Answer any *two* parts : 5×2=10

(a) Show that the function f defined on R by

$$f(x)=\begin{cases}1, & \text{if } x \text{ is rational}\\ 0, & \text{if } x \text{ is irrational}\end{cases}$$

is discontinuous at every point. 5

(b) Evaluate : 2+3=5

(i) $\lim_{x\to 0}\frac{e^{1/x}}{e^{1/x}+1}$

(ii) $\lim_{x\to 0}\frac{1-2\cos x+\cos 2x}{x^2}$

(c) Prove that a continuous and strictly increasing function f in $[a, b]$ is invertible and the inverse function is continuous in $[f(a), f(b)]$. 5

(d) Show that

$$\frac{\tan x}{x}>\frac{x}{\sin x} \text{ for } 0<x<\frac{\pi}{2}$$ 5

7. Answer any *two* parts : 5×2=10

(a) If f is derivable at c and $f(c)\neq 0$, then prove that the function $\frac{1}{f}$ is also derivable thereat, and then obtain the result

$$\left(\frac{1}{f}\right)'(c)=-\frac{f'(c)}{\{f(c)\}^2}$$ 5

(b) State and prove Roll's theorem. 1+4=5

(c) If c is an interior point of the domain of a function f and $f'(c)=0$, then show that the function has a maxima or a minima at c, according as $f''(c)$ is negative or positive. 5

(d) Use Taylor's theorem with $n=2$ to approximate $\sqrt[3]{1+x}$, $x>-1$. 5

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