Logical Analogies: Interpretations, Oppositions, and Probabilism

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Abstract: I present two logical systems to show the "analogy of proportionality" common to several interpretations: modality (necessity and possibility), quantification, truth-functional relations, moral attitudes (deontic logic), states of knowledge (epistemic logic), and states of belief (doxastic logic). To display the two underlying analogical relations, I call upon the originally Scholastic convention, recently put to use again, of using squares, hexagons, and octagons "of opposition". A combined epistemic–deontic logic happens to be found in the traditional "probabilist" theory of the "good conscience", and I shall then briefly explain how this is so.

Keywords: analogy; modal logic; Scholasticism; square of opposition; hexagon; octagon; probabilism; Aquinas; Leibniz; Hintikka

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I present two logical systems that display an "analogy of proportionality" among six areas of interpretation discussed by logicians today and by the Schoolmen of the past: necessity and possibility, quantification, propositional connectives, moral attitudes (deontic logic), and states of knowing and believing (epistemic and doxastic logics). To illustrate the underlying relations, I call upon the convention—Scholastic in origin but recently revived—of using figures "of opposition": squares, hexagons, and octagons. A combined epistemic–deontic logic happens to be found in the traditional "probabilist" debate over the good conscience, and I shall briefly explain how this is so[1].

1. The Logical Systems

The two systems, "Σ1 and "Σ2", can be represented by a hexagon and an octagon, respectively. Σ2 contains (logically implies) Σ1. Each system has the same two monadic operators, one strong (S) and one weak (W), governing propositional expressions like "p" or "q" ("Sp", "Wp"). The operators are interdefinable:

$$Sp =_{df} {\sim}W{\sim}p$$
$$Wp =_{df} {\sim}S{\sim}p$$

The Scholastic "square of opposition" displays the basic logical relations among the propositions (Figure 1, the traditional names are italics):

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[1] I use the following propositional symbols: ~p (negation: "not p"), p⊃q (implication: "if p then q"); p∧q (conjunction: "p and q"); p∨q (disjunction: "p or q"); p≡q (equivalence: "p if and only if q"); p | q (non-conjunction: ~[p∧q]: "not: p and q"); p≈q (contradiction or non-equivalence: ~[p≡q]: "if p not q and if q not p"). Quantifiers: ∀xϕx: "for all x, phi-x" and ∃xϕx: "there is at least one x such that phi-x" (where ϕx is a sentence containing x). Modal symbols: □p: "it is necessary that p" and ◇p: "it is possible that p".
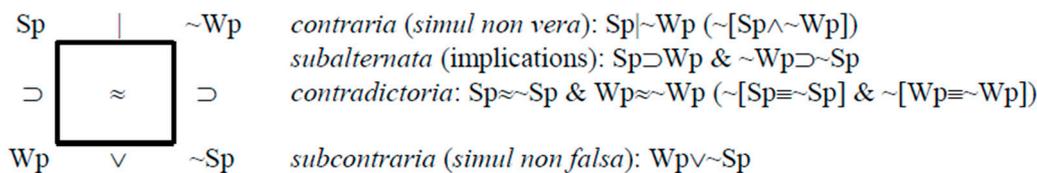
Figure 1. The Scholastic "square of opposition".

Since the operators are interdefined, the square may be expressed with either operator:

Sp S~p　　　　　　~W~p ~Wp
~S~p ~Sp　　　　　Wp W~p

The first logical system, Σ1, is illustrated by a hexagon, which incorporates the square but adds the disjunction of the contraries Sp∨~Wp, and the conjunction of the subcontraries Wp∨~Sp (Figure 2). The second logical system, Σ2, contains the first, but adds implications to and from propositions not governed by the operators S and W ("p" and "~p" alone), which constitute, as it were, an "outside link" to the network of propositions affected by the operators:

Sp⊃p　　　　　　~Wp⊃~p
p⊃Wp　　　　　　~p⊃~Sp
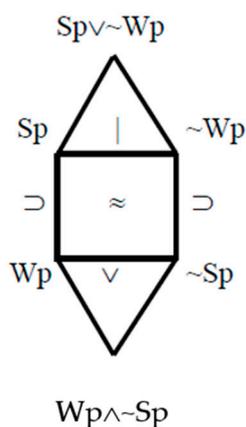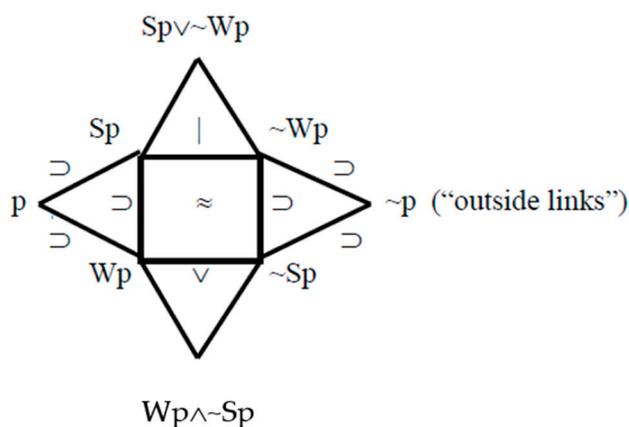
Figure 2. System Σ1 and System Σ2.

2. Modality

A basic interpretation of these systems is (alethic) *modality*, which includes the notions of necessity, possibility, contingency, actuality, and their negations. Here, the strong operator indicates *necessity*: □p ("it is necessary that p") and the weak operator *possibility*: ◇p ("it is possible that p"). *Contingency* is the conjunction of the subcontraries: ◇p∧◇~p ("it is possible that p and it is possible that not-p") or equivalently ◇p∧~□p ("p is possible but not necessary"). The disjunction of the contraries, □p∨~◇p, represents non-contingency ("p is either necessary or impossible"). Also, since necessity implies actuality and actuality implies possibility,

□p⊃p　　　　　　□~p⊃~p
p⊃◇p　　　　　　~p⊃◇~p,

this "outside" link to actuality is expressed in an octagon representing system Σ2 (Figure 3):
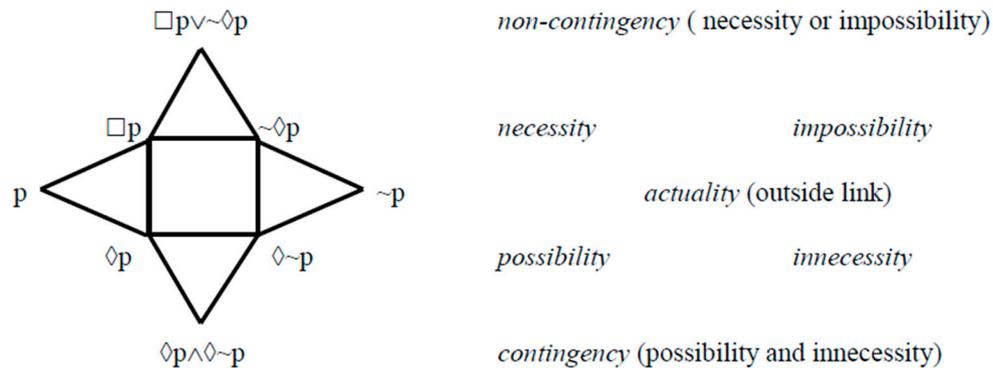
Figure 3. System Σ2.

This system, often called "T", is a basic modal system. Because the operators are interdefinable, it can be expressed in terms of necessity or of possibility,

$$\begin{array}{llll}
\Box p & \Box{\sim}p & {\sim}\Diamond{\sim}p & {\sim}\Diamond p \\
{\sim}\Box{\sim}p & {\sim}\Box p & \Diamond p & \Diamond{\sim}p.^2
\end{array}$$

3. Propositional Relations

The Schoolmen also used a square of opposition to illustrate the relations among propositions connected by "and" (∧) and "or" (∨). The strong operator S here represents *conjunction*: p∧q ("p and q") and the weak operator W represents (inclusive) *disjunction* p∨q ("p or q"). The relations are between the "molecular" propositions p∧q and p∨q. Σ2 is required, since there is an outside link to "atomic" propositions, those not joined by these connectives. The following implications indicate the outside link:

[p∧q]⊃p and [p∧q]⊃q　　　　　　　　　　[~p∧~q]⊃~p and [~p∧~q]⊃~q
p⊃[p∨q] and q⊃[p∨q]　　　　　　　　　　~p⊃[~p∨~q] and ~q⊃[~p∨~q]

The propositional octagon (Figure 4):
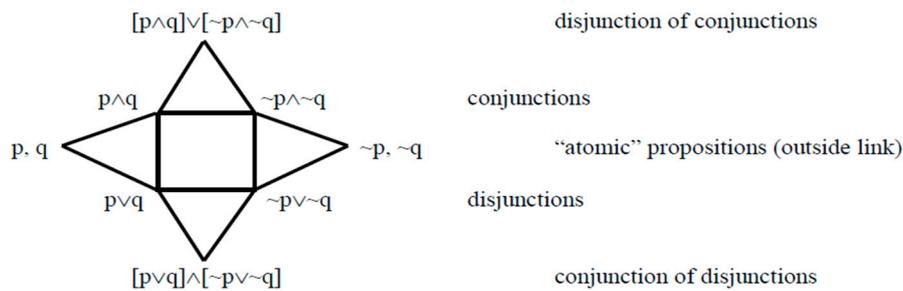
Figure 4. The propositional octagon.

4. Quantification

The square of opposition has also been applied to quantified propositions. Here, we may read the logical operators as quantifiers; the strong operator S is the *universal quantifier*: "∀x" and the weak operator W is the *existential quantifier*; "∃x". The letter "a" stands for a "thing" and the Greek letter phi (φ) indicates a proposition containing a (φa; say, "Alice is good"); the letter "x" is a variable in the proposition (φx). Again, Σ2 is needed, since the rules of quantificational logic,

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2　　Hughes and Cresswell [1], pp. 30ff.

$$\forall x \phi x \supset \phi a \qquad\qquad\qquad \forall x \sim\phi x \supset \sim\phi a$$
$$\phi a \supset \exists x \phi x \qquad\qquad\qquad \sim\phi a \supset \exists x \sim\phi x,$$

constitute an outside link. Here, connections to and from propositions are not governed by a quantifier[3].

The octagon of quantification (Figure 5):
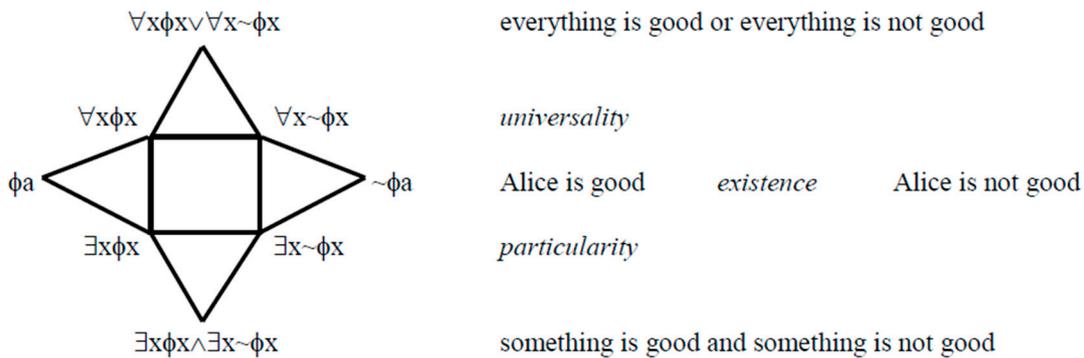
Figure 5. The octagon of quantification.

5. Deontic Logic

"Deontic logic" is the logic of τὸ δέον, *debitum*: the "*ought*" ("should"), the logic of "duty" or "obligation"[4]. Ways of signifying moral notions lie deeply embedded in our languages and can be formulated logically in systems compatible with various ethical views. The Schoolmen touched upon the matter, Leibniz formulated a deontic system, and more recent systems date from G. H. Von Wright [3,4][5].

For the strong operator S in deontic logic, I use the letter "O" ("ought") in the proposition "Op", and for the weak operator W, I use the letter "M" ("may") in "Mp". Either symbol can be taken to relate a free moral agent, say, Socrates, to the state-of-affairs that ought to obtain (Op: "Socrates should bring it about that p"; and Mp: "Socrates may bring it about that p").

O and M, too, are interdefinable:

Op = $_{df}$ ~M~p　　　　Socrates should do p just in case he may not leave p undone
Mp = $_{df}$ ~O~p　　　　Socrates may do p just in case he is not obliged to leave p undone

"O~p" can be abbreviated as "Fp" ("Socrates is forbidden to do p"). Also, ~Op can be read "Socrates need not do p", abbreviated "Np"[6]. We have, then, these equivalences:

3　Traditionally, the square has been applied to general propositions governed by words like *omnis, aliquis, nullus, aliquis...non*, a in the square:

　　all　　　　　none
　　some　　　　some...not

"All horses whinny" is an example of a universal, affirmative proposition. However, if it is read "for anything: if it is a horse, then it whinnies", no inference to "some horse whinnies" can be made unless at least one horse is stated to exist. This so-called "Fregean" analysis has the advantage of separating a statement of relation and a statement of existence.
4　Plato spoke of "τὰ δέοντα πρὰττειν" [2] (*Charmides* 164 B) and "*debere*" in Latin was used of moral duty. The words "may", "must", and the like are obviously to be taken here only in the *moral* sense.
5　See Redmond 1998, 1999b and 2000, 1990 [5–9]; for the Scholastics, see [10], cap. 5; for Leibniz's "*modalia juris*" in his *Elementa juris naturalis* [11] see [12], pp. 420ss); see also G. H. Von Wright (1951) [3,4].
6　Note that the phrases "ought not", "should not", and "may not" (in my dialect of English) usually mean O~p rather than ~Op or M~p.

$$Op = {\sim}M{\sim}p$$
$$O{\sim}p = {\sim}Mp = Fp$$
$${\sim}Op = M{\sim}p = Np$$
$${\sim}O{\sim}p = Mp,$$

as well as the implications $Op{\supset}Mp$: (if Socrates must, he may) and $Fp{\supset}Np$ (if he is forbidden, he need not).

Scholastic moral theologians, in their "probabilist" discussions, of which we shall speak below, called the disjunction of the contraries "law (*lex*)", since Socrates *must* either do p or not do p. They called the conjunction of the subcontraries "freedom (*libertas*)", since Socrates *may*, but need not, do p (in other words, p is "optional", a "work of supererogation"). The hexagon of Σ1 displays these relations, since there is no outside link to actual behavior; the fact that p ought to be the case does not mean that it will be the case ($Op{\supset}p$ is obviously invalid, as is $Fp{\supset}{\sim}p$). Leibniz displayed these relations in a square of opposition, and I include his terms in italics (for him, "optional" was "indifferent"); I also include the Scholastic "law" and "freedom" (Figure 6).
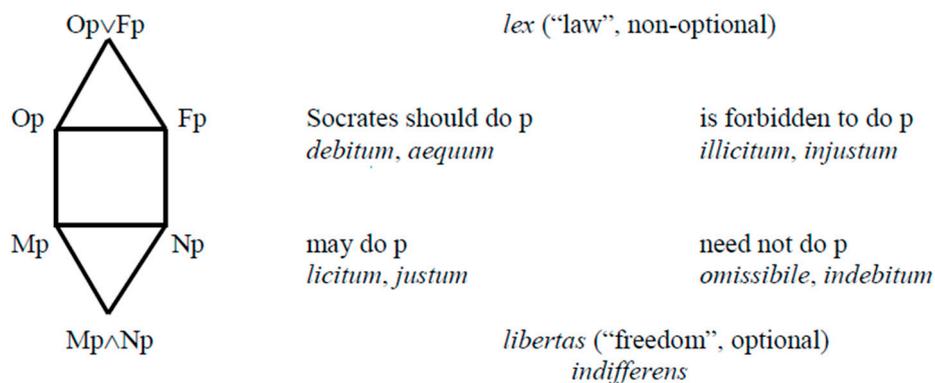
Figure 6. The square of opposition.

Again, we can express this system with a single operator:

Op	${\sim}Op$	${\sim}M{\sim}p$	${\sim}Mp$
${\sim}O{\sim}p$	${\sim}Op$	Mp	$M{\sim}p$.

Leibniz used his square in a legal context, but the Schoolmen were concerned with individual conscience, the "moral law".

6. Behavior

If we introduce a link to actual *behavior*, we have, as it were, systems of "virtue and vice". Socrates does his duty ($Op{\wedge}p$, $O{\sim}p{\wedge}{\sim}p$) or fails to do it ($O{\sim}p{\wedge}p$, $Op{\wedge}{\sim}p$)[7]. The schoolmen spoke of "*veritatem manifestare*"[8], and we can use an octagon to express the outside link to "compliance"; that is, the implications

$Op{\supset}p$	$Fp{\supset}{\sim}p$
$p{\supset}Mp$	${\sim}p{\supset}Np$.

The octagon of good behavior (Figure 7):

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7   Here, we must take "p" as "Socrates does p", so that Op should be read "it ought to be the case that Socrates does p".
8   Merkelbach [13], p. 63.
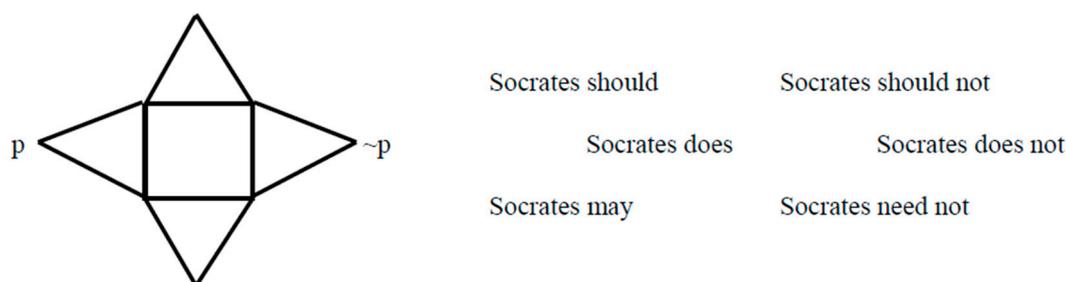
Figure 7. The octagon of good behavior.

For example, Socrates pits the olives when his wife Xanthippe told him to, and he does not eat all the olives when she told him not to (O p⊃p and Fp⊃~p). And, when he does eat all the olives, Xanthippe told him he might, and when he doesn't pit any olives, she said he doesn't have to (p⊃Mp and ~p⊃M~p).

On the other hand, we have an "octagon bad behavior" when p and ~p are switched, and, in the case of *lex*, compliance would become "non-compliance" (Figure 8):
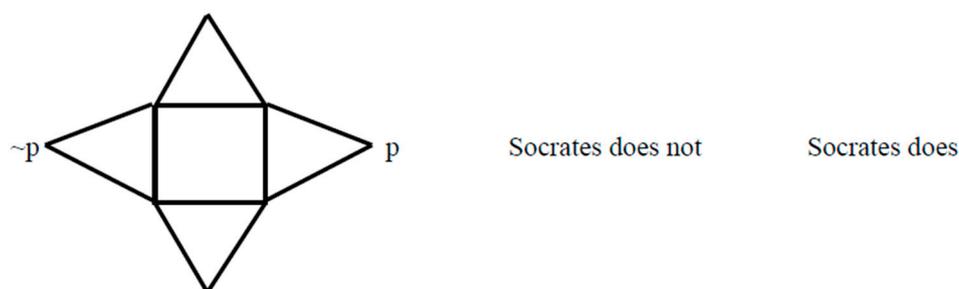
Figure 8. The octagon of bad behavior.

Here Socrates doesn't pit the olives when Xanthippe told him to, and he eats all the olives when she told him not to (Op⊃~p and Fp⊃p). But, in the case of *libertas*, when he does not eat all the olives, Xanthippe said he might, and when he pits the olives, she told him he didn't have to (~p⊃Mp and p⊃Np).

7. The Logic of Knowledge

Traditionally, good behavior requires that the person be able to act both *knowingly* and *freely*. If Socrates acts freely, he then meets the following condition, where "Dp" stands for "Socrates decides freely to bring about the state-of-affairs p" and lambda "λ" stands for "the physical laws of nature are in force"):

Dp∧◊[λ∧~Dp]

"Socrates decides to bring about p but it may be that while the laws of nature hold he does not so decide"[9].

Besides being free, Socrates must also *know* or at least *believe* that he has a duty. We may take *knowing* and *believing* in the sense of Jaako Hintikka's "logic of the two notions" [14][10]. In fact, the logics of *knowledge* and *belief* correspond to the two logical systems Σ2 and Σ1.

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[9]   Thus, the state-of-affairs p is not *physically* necessary; nor is it therefore logically necessary (◊[λ∧~p]⊃~□p).
[10]   Hintikka 1962, Section 1.1. "Epistemic" refers to knowing, although it is used in a general sense to include believing; "doxastic" refers only to believing.

In Hintikka's epistemic logic, the operator "K" relates a knower (say, Socrates) to what he knows, the state-of-affairs p in "Kp", meaning "Socrates knows that p"[11], This means that Socrates *knows* that p, where the knowledge is truth-entailing, that is:

* Kp⊃p and K~p⊃~p	what Socrates knows is true
* p⊃~K~p and ~p⊃~Kp	he does not know what is false.

We have, then, an outside link to "truth", to what is actually true or false "outside knowledge". In the following octagon, Kp∨K~p means that Socrates knows *whether* p is true or false, and ~K~p∧~Kp that he does *not* know whether p is true or false (Figure 9).

Figure 9. The octagon of knowledge and ignorance.

The octagon with the deontic operator O (Figure 10, in terms of Socrates's duty in regard to p):

Figure 10. The octagon with the deontic operator O.

8. The Logic of Belief

In Hintikka's doxastic logic, the operator "B" relates Socrates to his opinion, that is, the state-of-affairs p in "Bp" ("Socrates believes that p"). "Belief", in this sense, presupposes basic rationality and does not have the negative connotations that the word may have in a Platonic tradition (δόξα, *opinio*). Nevertheless, belief, although it follows from knowledge (Kp⊃Bp), is not truth-entailing (Bp⊃p and Bp⊃Kp are invalid). Therefore, there is no outside link, and Σ1 must be used to display the relation (Figure 11):

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[11] We need not use Hintikka's symbol "a" for the epistemic agent. He uses the letters "P" and "C" to indicate the weak counterparts of Kp and Bp (~K~p and ~B~p), which he glosses as "it is possible, for all (Socrates) knows, that p" and "it is compatible with everything that (Socrates) believes that p".

Figure 11. Σ1 with the operator B.

9. The Logic of Knowledge and Belief Combined

Since epistemic logic contains the doxastic logic, we may combine the octagon of knowledge with the hexagon of belief. The double lines are implications linking the two systems:

$Kp \supset Bp$ $\quad\quad\quad\quad\quad\quad$ $K\sim p \supset B\sim p$

$\sim B\sim p \supset \sim K\sim p$ $\quad\quad\quad\quad$ $\sim Bp \supset \sim Kp.$

The diagram (Figure 12):
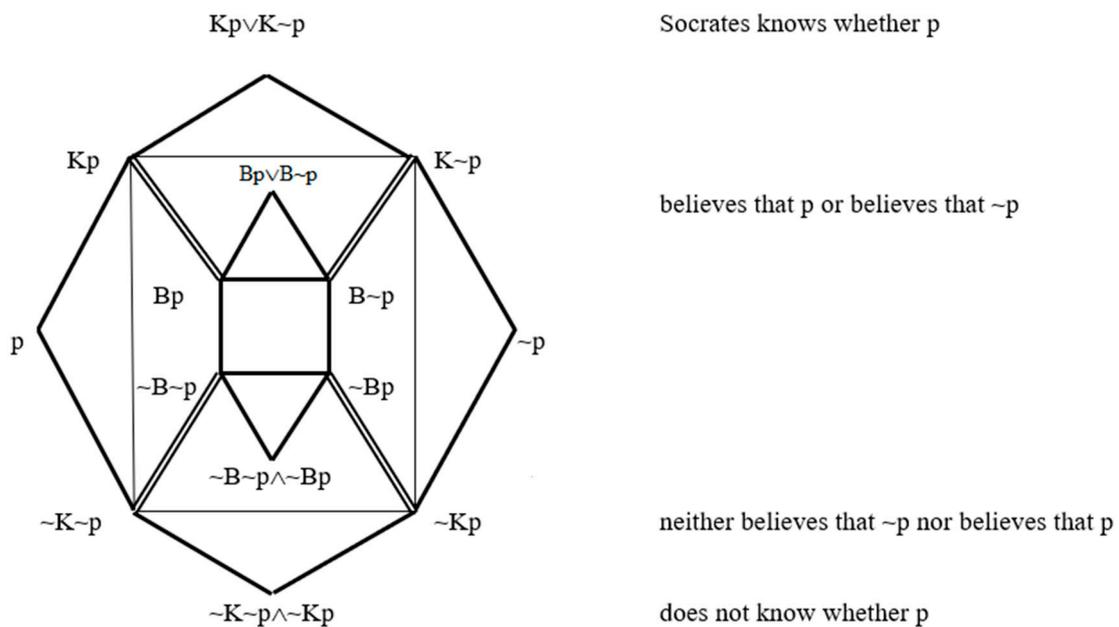
Figure 12. The octagon of knowledge with the hexagon of belief combined.

This network of implications runs from "certitude" to "quandary"[12].

10. Probabilism

The Schoolmen around the 17th century passionately discussed how we are to judge when the moral agent is bound by *law* or has the *liberty* to act as he pleases. The purpose of this metaethical controversy, called "probabilism"[13], was to describe the "good conscience", particularly in the face of

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12 Also, $[Kp \lor K\sim p] \supset [Bp \lor B\sim p]$ and $[\sim B\sim p \land \sim Bp] \supset [\sim K\sim p \land \sim Kp]$.

13 The word was first used by the Spanish Dominican Bartolomé de Medina (1527–1580).

moral dilemma. A number of theories or *systemata* ("systems") developed, and they all relied upon a calculus of *thinking about duty*—combining, in effect, an epistemic logic with a deontic logic.

Obviously, we *may* do something when we are *sure*, that is, when we *know*[14] that we may (KMp⊃Mp; also, KFp⊃Fp) or even if we are unsure but *believe* that we may (BMp∧~KMp). But what do we do if we "waver": neither believing that the act is forbidden nor that it is allowed (~BFp∧~BMp)? Or how are we to act when ethicists themselves have sharply different views on what is right?

Participants in the discussion distinguished several "states of mind" according to the degree of rationality of these states.[15] They took this rationality as a "probability of grounds" of the reasons or arguments brought to defend their positions. "Probable" here does not mean "likelihood" (say, of heads or tails turning up at the toss of a coin); a "probable" argument is rather one that "can be defended by sound arguments (*rationes graves*), and is quite close to Hintikka's notion of "defensibility"[16].

These are the chief states of moral thinking[17]:

* *Certainty*: Socrates is *sure* that he must or may effect the state-of-affairs p if he has a "sound grounds" for his attitude (KOp, KMp or their negative forms) and no reason, or only "slight (*levis*)" ones, for the opposite. In this state truth would be "manifested" (KOp⊃Op, KMp⊃Mp). He then knows *whether* he may or may not act:

    KMp∨KFp

* *Opinion*: Socrates *believes* that he should or may effect p when his reasons are "sound enough" or "sounder than the opposite". "And so, although the truth is not "manifested, still there is a notable appearance of the truth"[18]. Hence, he *opines* that some act is licit or forbidden:

    BMp∨BFp

    and he is then said to be "afraid of being wrong".

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[14] On being certain and knowing, see Hintikka Sections 5.8 and 5.9.

[15] Merkelbach, 62. The doctrine of St. Thomas Aquinas ([15], 2–2:1–16.) lies in the background. He distinguished among knowing, opining, suspecting, and wavering (*scire, opinari, suspicari, dubitare*) according to assent to a proposition (*assensionem*) and adherence (*adhaesio*) to its truth-value. His "epistemic calculus" can be approximately described with the help of the operators Kp and Bp (Figure 13):

	p						**~p**
wavering				~B~p∧~Bp			
suspecting			~B~p		~Bp		
opining		Bp				B~p	
knowing	Kp						K~p
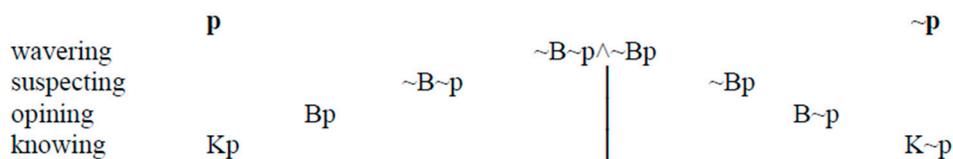
Figure 13. St. Thomas Aquinas's "epistemic calculus".

For Thomas faith is truth-entailing but knowledge-eschewing, for when someone assents to p on religious grounds, then p, but also Kp. Hintikka calls such a state not logically, but "epistemically", inconsistent (K[p∧~Kp] is a theorem of the epistemic system). It is also, like "Moore's paradox" (p∧~Bp), doxastically inconsistent (B[p∧~Bp]). Cf. Hintikka pp. 71ff and 79ff, Moore [16] p. 125; [5,9].

[16] Hintikka, Section 2.6. Any argument, not only moral, can be "probable". The concept goes back to Aristotle's "dialectical" arguments: ἐξ ἐνδόξων: those that seem true "to all or most or the wise; to all or most of them or to the best known or esteemed", *Topica* I (100B22) [17].

[17] Merkelbach, 62–63. These "systems" (with their variations) are highly nuanced, and our symbolic rendering is only approximate.

[18] Merkelbach, 63–4; vide Redmond 1998. There are degrees of the probable: "truly and solidly" (probable), "quite", "more", "much more (or less)", "most probable", "equally probable", ... (ibid. 66–67); a "fuzzy logic" would represent these differences.

* *Wavering*[19]: Socrates wavers "positively" when he suspends assent because the reasons for both positions (Fp and Mp) are "equally sound". Hence in wavering, he does not believe that he may not, nor does he believe that he may:

  ~B~Mp∧~BMp.

* *Ignorance* or "negative wavering": Socrates suspends his assent when his reasons for both sides ("parts") are totally lacking. He therefore *does not know whether* he may or may not act:

  ~K~Mp∧~KMp.

The similarity between the four Scholastic and the "epistemic states" is obvious and can be compared in this diagram, which combines epistemic, doxastic, and deontic logics (Figure 14):
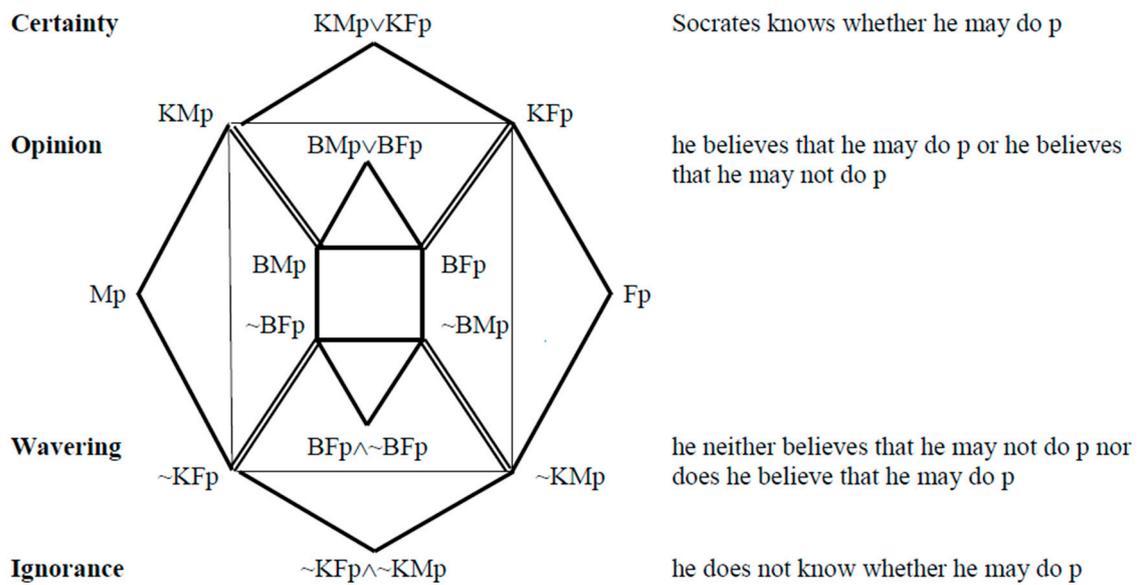
Figure 14. Epistemic, doxastic, and deontic logics combined.

The double lines connect knowledge with belief:

* that Socrates knows that he may do p implies that he believes that he may: KMp⊃BMp (also KFp⊃Bfp);

* that he does not believe that he may not do implies that he does not know that he may not: ~BFp⊃~Kfp (also ~Mp⊃~KMp).

These are the (epistemic) truth-entailing relations:

* that Socrates *knows* that he may do p implies that he may: KMp⊃Mp (or KFp⊃Fp);

* that he *may* do p implies that he does not *know* that he is forbidden: Mp⊃~KFp (or Fp⊃~KMp).

The various "systems" lie on a spectrum between the extreme positions called *rigorism* or *tutiorism* (a moral act demands certitude) and *laxism* (any opinion, even a doubtfully sound one, may be followed)[20]. Other systems are *mitigated tutiorism* (the soundest view is always to be followed), *probabiliorism* (the sounder view is to be followed), *aequiprobabilism* (a position at least as sound as its opposite may be followed), and *moderate probabilism* (a solidly probable viewpoint may be followed).

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[19] "Suspicion" is included in wavering, for "reason accepts what is little as if it were nothing" (quod parum est quasi nihil accipit ratio), [15] (1–2:14:4); Merkelbach, 63.

[20] Both positions were condemned by the Catholic church (Merkelbach, 70 and 72). For a more detailed description see Redmond 1998. The probablist did not lack humor; the Cistercian John Caramuel was called the "Prince of the Laxists", and another laxist, Antonio Diana, a Theatine priest, was said to be "the lamb who taketh away the sins of the world" (agnus qui abstulit peccata mundi)!

11. Analogy

In summary, in an "analogy of proportionality", there are a number of interpretations of the systems Σ1 and Σ2 according to their strong and weak operators (Figure 15):

$$\frac{strong}{weak} : \frac{necessary}{possible} = \frac{all}{at\ least\ one} = \frac{and}{or} = \frac{should}{may} = \frac{know\ that}{not\ know\ that\ not} = \frac{believe\ that}{not\ believe\ that\ not}$$

Figure 15. Interpretations of the systems Σ1 and Σ2.

All these "ratios" embody Σ1, and Σ2 also applies to those having an "outside link": *necessary* (the link to actuality), *all* (to existence), *and* (to atomic propositions), and *knowing* (to truth). There is also an analogy between the epistemic, doxastic, and deontic logics described and those found in Scholastic probabilism.

Conflicts of Interest: The author declares no conflict of interest.

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