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Entanglement Dynamics of Three and Four Level Atomic System under Stark Effect and Kerr-Like Medium

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Abstract: We investigated numerically the dynamics of quantum Fisher information (QFI) and entanglement for three- and four-level atomic systems interacting with a coherent field under the effect of Stark shift and Kerr medium. It was observed that the Stark shift and Kerr-like medium play a prominent role during the time evolution of the quantum systems. The non-linear Kerr medium has a stronger effect on the dynamics of QFI as compared to the quantum entanglement (QE). QFI is heavily suppressed by increasing the value of Kerr parameter. This behavior was found comparable in the cases of three- and four-level atomic systems coupled with a non-linear Kerr medium. However, QFI and quantum entanglement (QE) maintain their periodic nature under atomic motion. On the other hand, the local maximum value of QFI and von Neumann entropy (VNE) decrease gradually under the Stark effect. Moreover, no prominent difference in the behavior of QFI and QE was observed for three- and four-level atoms while increasing the value of Stark parameter. However, three- and four-level atomic systems were found equally prone to the non-linear Kerr medium and Stark effect. Furthermore, three- and four-level atomic systems were found fully prone to the Kerr-like medium and Stark effect.

Keywords: quantum Fisher information (QFI); non-linear Kerr effect; Stark effect; three- and four-Level atomic system

1. Introduction

Parameter estimation has diverse applications in the field of quantum information technology. The newly developed methods and techniques of measurements to assess the sensitivity of the parameters involved have led to scientific revolutions and advancement in this key technology. A great deal of work and research has been done on phase estimation related to the practical problems of state generation, loss and decoherence [1–3]. Fisher information (FI) lies at the heart of parameter estimation theory that was originally introduced by Fisher in 1925. Fisher information is considered as a key concept in quantum estimation and quantum information theory [4,5]. The Fisher information describes the sensitivity of a state with respect to perturbation of the parameters. The larger is the FI, the higher would be the precision of estimation. The extension of the FI to the quantum regime, i.e., the quantum Fisher information (QFI), allows the characterization of information in quantum domain [6,7], which can be explained by the quantum Cramer–Rao bound [8]. Therefore, to improve the precision, quantum resources such as coherence or entanglement can be used to characterize and manipulate the quantum state of a system with high precision.

Quantum entanglement (QE) is a very famous and mysterious phenomenon of quantum mechanics that cannot be described completely. It has different applications in quantum mechanics and quantum information theory. Quantum entanglement (QE) was first studied by Schrodinge [9,10] as a basic phenomenon of quantum mechanics and it has no similarity with classical approach [11]. On the other hand, quantum correlations are used to calculate the quantum states of complex systems. The correlations of complex systems do not depend on the spatial separation of components, so the system behaves as a single system. Schrodinger explained that the information of different parts of the system would not contain the complete information of the whole system [11]. Thus, quantum correlations are the result of the quantum measurements that explain the solution and information of different physical systems such as Bell inequalities [12,13] and confirm the experimental segment of the spooky action at a distance. During the last few years, due to the extensive progress in the field of quantum information processing (QIP), the new field of quantum metrology has become important and prominent [14,15]. The importance of QE in a different process has led to the investigation of larger dimensional quantum systems and has shown an important and significant role in quantum systems of many particles [16]. Since the quantum systems are not completely closed, the dynamical response is observed when the system loses coherence due to interaction with the environment. During the interaction of a quantum system with the environment, the dynamics of the system behave as an open system. Hence, the study of the dynamics of different physical quantities, during the interaction between complex quantum systems and the environment, becomes very attractive and interesting. This interaction causes a quantum noise that creates fluctuations such as decoherence and dissipative dynamics that are not reversible [17]. Therefore, in the last few decades, the study of quantum decay caused by the interaction with the environment has attracted much attention. Moreover, the dynamics of open quantum systems has received numerous attention in the field of modern quantum theory, rather more precisely to the field of quantum information processing.

The Jaynes-Cumming model (JCM) [18] represents the interaction between a two-level atom and cavity field of a quantized single-mode under the rotating wave approximation (RWA). It is the simplest basic model of matter–field interaction having a precise integrable Hamiltonian. The experimental studies of Rydberg atoms confirm the predictions of the JCM [19–21]. Furthermore, as an improvement in the study of JCM, its different generalizations such as intensity dependent JCM and multi-photon generalizations of JCM [22,23] have also been introduced. Moreover, the interactions of a moving four-level N-type atom with a three-mode cavity field in the presence of a nonlinear Kerr-like medium were investigated by F An-Fu, W Zhi-Wei [24].

A Kerr-like medium can be represented by a harmonic oscillator (HO) [25]. The non-linearity of the Kerr-like medium is similar to a Hamiltonian that is quadratic in the number of photon operator. The Kerr- like medium model can be presented as if the cavity is having two different types of Rydberg atoms, one of them behaves as harmonic oscillator and the other one is coupled to the field-mode. The effect of non-linear interaction of a cavity mode with a Kerr-like medium on the dynamics of atomic population has been studied by taking into account the non-linear generalization of the JCM [26–28], where the interaction is mainly confined to the case of single cavity mode. Quantum dynamics of a two-level atom having interaction with two-/multi-modes of the cavity field in the presence of a Kerr like medium has also been investigated [29–31]. Berlin and Aliaga [21] presented the case of the non-degenerate two-photon JCM model in the presence of a Kerr-medium using a method of operators that are related to the dynamics of the system and investigated the temporal evolution of the system.

Recently, the JCM model was extended to consider the atomic motion along the axis of the cavity by Schlicher [32]. The extended JCM contains external atomic effects because the atomic motion is quantized and the motion of the center of the mass of an atom is cooled down to very small temperatures, hence the vibrational motion becomes quantized [33]. The properties of the atomic external and internal quantities such as the radiation force and momentum of the atom for the Raman-coupled JCM have also been investigated [33,34]. The interaction of a moving three level atom with two-mode cavity field including a Kerr-like medium can be seen in references [35,36]. Abdel-aty [37] studied the quantum entanglement of a three-level atom interacting with two-mode cavity in a nonlinear medium. The studies in this direction also consider the atoms initially prepared in the eigenstate of momentum and the state of the field being squeezed. Abdel-Wahab [38] considered the atomic system containing a moving Rubidium atom interacting with a single-mode cavity field. Moreover, the resonant case of the interaction of a moving N-level ladder type atom with cavity field of $(N-1)$-mode was observed by the same author [39] and the system having a four-level ladder type atom in a momentum eigenstate interacting with a single-mode cavity field under the influence of nonlinearities of the intensity-dependent coupling [40].

The interaction of light with matter is known as Stark effect and it is considered as an important important phenomenon in the field of quantum optics [41]. There is a splitting and shifting of spectral lines of atoms under the effect of electric field in this process. The Stark effect is investigated for the Dicke Hamiltonian in the presence of constant fields, which results in a shifting of the eigenvalues due to the emitter–cavity interaction strength. The dynamic Stark effect is observed in an optical system controlled by a laser beam. In the presence of the dipole approximation, there is an exact contribution of time to the Dicke Hamiltonian. Due to the interaction with external field, it is periodic with the frequency of the laser. The shift in the Rabi splitting is observed due to the dynamic Stark effect. The recent progress in the field of cavity quantum electrodynamics (QED) made it possible to achieve strong and ultra-strong light–matter-interaction patterns experimentally [42,43]. The full Dicke Hamiltonian drawing on the outer laser can be controlled to represent the dynamical properties. The Stark shift explains the multi-photon transition in a more precise and accurate manner. The significance of this process can be noticeable when two levels of atom interact with the middle level [44]. The Stark shift mechanism is explained in this situation by considering the process of adiabatic elimination of the middle level(s) of a multi-level atom [45]. This method calculates a simple effective Hamiltonian of a two-level atom interacting with a quantized field of single-mode with at least two-photon transition in the presence of Stark shift [46]. The Stark shift in two-photon transitions is taken as the detuning depending on the intensity [47]. The interaction of two two-level atoms and a single-mode field with degenerate two-photon transition in the presence of Stark shift was studied by Obada et al. [48] and it is investigated that the degree of QE increases with the rise in the value of the Stark shift parameter.

Von Neumann entropy is considered as the main entanglement measure for pure states. The correlation between quantum entanglement and Fisher information (FI) about a certain parameter in a quantum state has not been studied widely. Moreover, observations have been made to calculate the QE of pure state by using FI. The entanglement evaluation and QFI of N-level atomic system under the influence of intrinsic decoherence was studied by Jamal and co-workers [49]. S. Abdel-Khalek studied the monotonic relation between QFI and VNE for three-level moving atomic system interacting with single mode coherent field [50]. The time evolution of the QFI of a system whose dynamics is described by the phase-damped harmonic model was studied by Berrada et al [51]. It is observed that there is an interesting monotonic relation between the QFI and nonlocal correlations measured by the negativity depending on choosing the estimator parameter during the time evolution of the quantum system.

In this work, we considered a three-level atom in cascade type configuration with 0 as excited state, 2 as ground state and 1 as the intermediate level and a cascade-type four-level atom with upper state 0, ground state 3 and two intermediate states 1 and 2. These three- and four-level atoms are surrounded by a non-linear Kerr medium in an optical cavity. Our main focus was studying the interaction of electromagnetic field with three- and four-level atoms in a non-linear Kerr medium by including the stark shift term in the Hamiltonian as well. We calculated the QFI and VNE of the three- and four-level atomic systems interacting with coherent field under the effect of non-linear Kerr medium and Stark shift in the presence of atomic motion and without atomic motion. It was found that the three- and four-level atomic systems are equally prone to these environmental effects.

The paper is arranged as follows. In Section 2, we present the background related to VNE and QFI. The system Hamiltonian and the dynamics of three and four-level atomic system affected by Kerr medium and Stark shift are presented in Section 3. In Section 4, comprehensive results and numerical discussions are presented. In Section 5, we present a brief conclusion.

2. Introduction to Quantum Fisher Information (QFI) and Van Neumann Entropy

2.1. Quantum Fisher Information (QFI)

The classical Fisher information (CFI) for any particular phenomenon with a one parameter θ that is unknown can be written as

$$I_{\Phi} = \sum_i p_i(\Phi)[\frac{\partial}{\partial\Phi}\ln p_i(\Phi)]^2, \tag{1}$$

where $p_i(\Phi)$ represents the probability density having significant and prominent influence on the fixed parameter with the outcome $\{x_i\}$ of measurement for a distinct observable X. The CFI can be described with the help of the inverse variance of the asymptotic normality of a maximum-likelihood estimator. The QFI has a very important and effective part in quantum metrology and the large precise measurement of an unknown parameter can be gained, which has an inverse relation to the quantum fisher information and is given as [52].

$$F_{\Phi} = \mathrm{Tr}[\rho(\theta)D^2], \tag{2}$$

where $\rho(\Phi)$ is the density matrix (DM) of the system, Φ is the parameter to be estimated, and D represents the quantum score (symmetric logarithmic derivative), which is written as

$$\frac{d\rho(\Phi)}{d\Phi} = \frac{1}{2}[\rho(\Phi)D + D\rho(\Phi)]. \tag{3}$$

Here, we considered three- and four- level atomic systems with the density operator $\rho(\Phi)$. The spectral decomposition of the DM is described as

$$\rho_{\Phi} = \sum_K \lambda_K |k\rangle\langle k|, \tag{4}$$

The QFI which is related to Φ for this DM, is represented by [53]

$$F_{\Phi} = \sum_k \frac{(\partial_{\Phi}\lambda_k)^2}{\lambda_k} + 2\sum_{k,k'} \frac{(\lambda_k - \lambda_{k'})^2}{(\lambda_k + \lambda_{k'})} |\langle k|\partial_{\Phi}k'\rangle|^2 \tag{5}$$

where $\lambda_k > 0$ and $\lambda_k + \lambda_{k'} > 0$. The first term in the above equation represents the CFI and the second term describes QFI. Therefore, wecan represent the QFI of a bipartite density operator $\rho_{AB.}$ which is related to Φ as [54]

$$I_{QF}(t) = I_{AB}(\Phi, t) = \mathrm{Tr}[\rho_{AB}(\Phi, t)D^2(\Phi, t)] \tag{6}$$

where $D(\Phi, t)$ is the quantum score [55] (the symmetric logarithmic derivative) that can be written as

$$\frac{\partial\rho_{AB}(\Phi, t)}{\partial\Phi} = \frac{1}{2}[D(\Phi, t)\rho_{AB}(\Phi, t) + \rho_{AB}(\Phi, t)D(\Phi, t)] \tag{7}$$

2.2. Von Neumann Entropy

Von Neumann entropy (VNE) is the most efficient and basic QE measurement tool when the quantum system is in the pure state, therefore VNE is used to measure entanglement between the field and the atom [56]. The VNE is presented in the form of eigenvalues of

$$S_A = -Tr(\rho_A \ln \rho_A) = -\sum_i r_i \ln r_i \tag{8}$$

where r_i represents the eigenvalues of the atomic DM

$$\rho_A = Tr_B\left(\rho_{AB}\right) \tag{9}$$

3. Model and Hamiltonian

The model considered in the first case consists of three- and four-level moving atoms, which are interacting with coherent field in the presence of non-linear Kerr-like medium. We also include the stark shift term in the interaction Hamiltonian in the second case. In Figures 1 and 2, moving three- and four-level cascade type atoms are shown in the presence of non-linear Kerr medium.

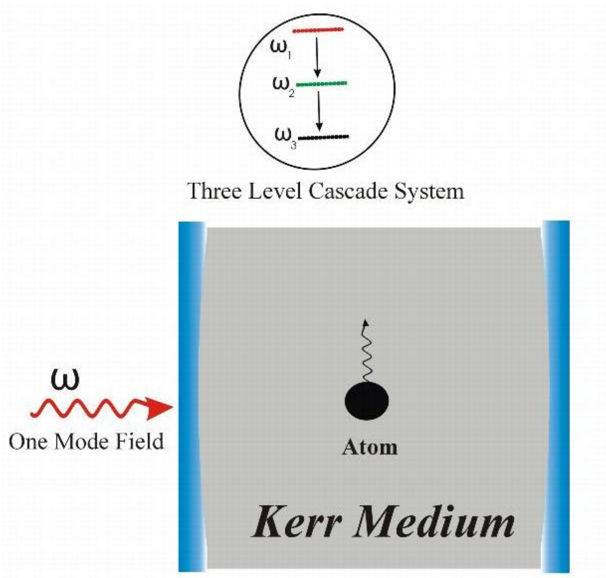

Figure 1. Moving three-level cascade type atom in the presence of non-linear Kerr medium.

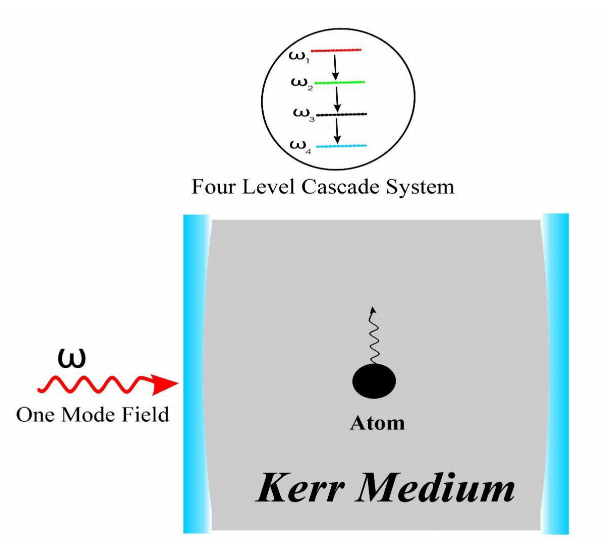

Figure 2. Moving four-level cascade type atom in the presence of non-linear Kerr medium.

The total Hamiltonian of the system $\hat{H}_T$ under the RWA for a particular system can be described as [57,58]

$$\hat{H}_T = \hat{H}_{Atom-Field} + \hat{H}_I. \tag{10}$$

where $\hat{H}_{Atom-Field}$ represents the Hamiltonian for the non-interacting atom and field, and the interaction part is given by $\hat{H}_I$. We write $\hat{H}_{Atom-Field}$ as

$$\hat{H}_{Atom-Field} = \sum_j \omega_j \hat{\sigma}_{j,j} + \Omega \hat{a}^\dagger \hat{a}, \tag{11}$$

where $\hat{\sigma}_{j,j} = |j\rangle \langle j|$ describe as population operators for the jth level. The interaction Hamiltonian of three- and four-level atomic systems for the non-resonant case can be written as [47]

$$\hat{H}_I = \sum_{s=1}^{N} \Omega(t) \left[\hat{a} e^{-i\Delta_s t} \hat{\sigma}_{s,s+1} + \left(\hat{a} e^{-i\Delta_s t} \hat{\sigma}_{s,s+1} \right)^\dagger \right]. \tag{12}$$

When $N = 2$, it is a three-level atomic system and, for $N = 3$, it represents the four-level atomic system.

In the case of non-linear Kerr-like medium, the interaction Hamiltonian can be written as

$$\hat{H}_I = \sum_{s=1}^{N} \Omega(t) \left[\hat{a} e^{-i\Delta_s t} \hat{\sigma}_{s,s+1} + \left(\hat{a} e^{-i\Delta_s t} \hat{\sigma}_{s,s+1} \right)^\dagger \right] + \chi \hat{a}^{\dagger 2} \hat{a}^2 \tag{13}$$

By including the Stark shift term, the interaction Hamiltonian becomes

$$\hat{H}_I = \sum_{s=1}^{N} \Omega(t) \left[\hat{a} e^{-i\Delta_s t} \hat{\sigma}_{s,s+1} + \left(\hat{a} e^{-i\Delta_s t} \hat{\sigma}_{s,s+1} \right)^\dagger \right] + \beta \hat{a}^\dagger \hat{a} |g\rangle \langle g|. \tag{14}$$

where $|g\rangle$ represents the ground state of atom. In the case of three-level atom $|g\rangle = |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and for four-level atom $|g\rangle = |3.\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and χ and β are the constants of Kerr-like medium and Stark effect. The detuning parameter is described as

$$\Delta_s = \Omega + \omega_{s+1} - \omega_s. \tag{15}$$

The coupling constant for atom and field is G, $\Omega(t)$ represents the shape function of the cavity-field mode [55] and the motion of atom is along z-axis. For particular interest

$$\begin{aligned} \Omega(t) &= G \sin(w\pi v t/L) \text{ in the presence of atomic motion }, w \neq 0 \\ \Omega(t) &= G \text{ in the absence of atomic motion } w = 0 \end{aligned} \tag{16}$$

where the velocity of motion of atom is v and w denotes half the number of wavelengths of the mode in the cavity and L describes the length of cavity along z-direction. Now,, take the velocity of atom as $v = \lambda L/\pi$ which gives us

$$\Omega_1(t) = \int_0^t \Omega(\tau) d\tau = \frac{1}{w}(1 - \cos(w\pi v t/L) \text{ for } w \neq 0 \tag{17}$$

$$= Gt \text{ for } w = 0. \tag{18}$$

To find the phase shift parameter as precisely as possible, we consider the optimal input state as

$$|\Psi(0)\rangle_{\text{Opt}} = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \otimes |\alpha\rangle \tag{19}$$

where $|1\rangle$ and $|0\rangle$ describe the states of atom and α is the coherent state of the input field given as

$$|\alpha\rangle = \sum_{n=0}^{\infty} \alpha^n \sqrt{e^{-|\alpha|^2}/n!}|n\rangle. \tag{20}$$

We consider a single-atom phase gate that introduces the phase shift as

$$\hat{U}_\phi = |1\rangle\langle 1| + e^{i\phi}|0\rangle\langle |0|, \tag{21}$$

$|\Psi(0)\rangle$ is obtained from the operation of the single-atom phase gate on $|\Psi(0)\rangle_{\text{Opt}}$

$$\hat{U}_\phi|\Psi(0)\rangle_{\text{Opt}} = |\Psi(0)\rangle \tag{22}$$

$$= \frac{1}{\sqrt{2}}(|1\rangle + e^{i\phi}|0\rangle) \otimes |\alpha\rangle \tag{23}$$

After the operation of phase gate, the system will interact with a field. The precision of the estimation is strongly affected by the characteristics of the interaction between the field and moving N-level atomic system. Now, we can write the state $|\Psi(0)\rangle$ as

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(|1, n+1\rangle + e^{i\phi}|0, n\rangle) \tag{24}$$

where $|1, n+1\rangle$ and $|0, n\rangle$ are allowable atom-field states. The state in which a number of photons are consistent with the atomic level are known as allowable atom field states. For N-level atomic system, the allowable basis are given by

$$|0, n\rangle, |1, n+1\rangle, |2, n+1\rangle, ...|N-2, (n+N-2)\rangle, |N-1, (n+N-1)\rangle, \tag{25}$$

where n are number of photons initially present in the cavity and $N-1$ are the number of levels in an atom. 0 and $N-1$ represent the excited and ground state of the atom. For the time-independent case, the wave function can be characterized by the transformation matrix $\hat{U}\,(t)$ as given by

$$|\Psi(t)\rangle = \hat{U}\,(t)\,|\Psi(0)\rangle, \tag{26}$$

where $\hat{U}(t)$ is given as

$$\hat{U}(t) = \sum_{z=1}^{3,4} \exp(-iE_z t)|\varphi_z(t)\rangle\langle\varphi_z(t)|, \tag{27}$$

where $|\varphi_z(t)\rangle$ and $E_z(t)$ are eigenvectors and eigenvalues of the Hamiltonian H_I, respectively for three and four-level atomic systems. The quantum Fisher information is calculated numerically for the atom-field density matrix given below

$$\hat{\rho}(t) = \hat{U}\,(t)\,\rho(0)\hat{U}^{\dagger}\,(t)\,. \tag{28}$$

The influence of the parameters w, ϕ, χ and β on the evolution of the QFI and VNE is presented in the next section.

4. Discussion of Numerical Results

In this section, we discuss the results of the numerical calculations for the time evolution of QFI and von Neumann entropy of three- and four-level atoms interacting with a coherent field under the influence of Kerr-like medium and Stark shift. For the sake of simplicity, we scaled out the time t, i.e., one unit of time is described by the inverse of the coupling constant G. Initially, we investigated the time evolution of QFI and von Neumann entropy for three- and four-level atomic systems interacting with a non-linear Kerr medium with and without atomic motion. In Figures 3 and 4, we plot QFI and von Neumann entropy (VNE) as a function of time for three- and four-level atoms interacting with coherent field under the influence of a non-linear Kerr medium for $|\alpha|^2 = 6$, $\chi = 0.3, 1, 3$, phase shift $\phi = 0$, $\pi/4$ and atomic motion parameter $w = 0$. It was observed that quantum entanglement is strongly influenced under the effect of non-linear Kerr medium as we increase value of the Kerr parameter χ. It was found that at, $\chi = 0.3$, QFI decreases more rapidly as compared to the VNE for $\chi = 1, 3$. The behavior of three- and four-level systems was found comparable in a non-linear Kerr medium. However, a monotonic relation was observed for both for QFI and VNE in the case of no atomic motion. This shows the strong dependence of QFI and entanglement on the Kerr-like medium. In Figures 5 and 6, we plot the QFI and von Neumann entropy as a function of time for the system of three- and four-level atoms having interaction with coherent field in the presence of non-linear Kerr effect for $|\alpha|^2 = 6$, $\chi = 0.3, 1$, 3, phase shift $\phi = 0$, $\pi/4$ with atomic motion, i.e., $w = 1$. A periodic behavior of QFI and entanglement was observed in the presence of atomic motion. It was also observed that the atomic motion enhances the periodicity of oscillations. Furthermore, for $w = 1$, entanglement death and re-birth was observed. In Figures 7–10, we plot QFI and von Neumann entropy as a function of time for three- and four-level atomic systems interacting with coherent field for $|\alpha|^2 = 6$, $\beta = 0.3, 1, 3$, phase shift parameter $\phi = 0$, $\pi/4$ and atomic motion parameter $w = 0$ and 1, respectively. In the presence of Stark effect, the local maximum values at the revival time of both QFI and VNE decrease gradually during the time evolution. However, the behavior of QFI and QE is similar both for three- and four-level systems under the Stark effect. Finally, we conclude that the three- and four-level atomic systems are fully prone to the Kerr-like medium and Stark effect.

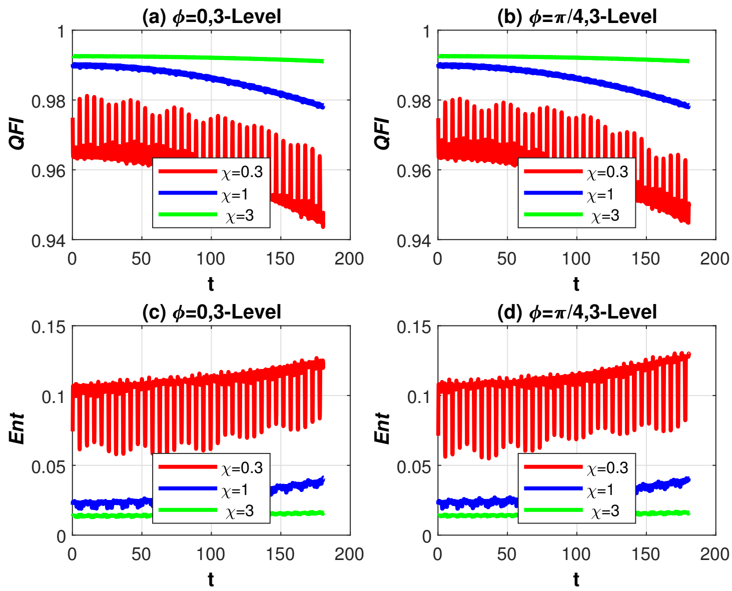

Figure 3. (Color online) The QFI (**top**) and von Neumann entropy (**bottom**) as a function of time for a system of three-level atom having interaction with coherent field for $|\alpha|^2 = 6$ and the phase shift estimator parameter $\phi = 0$ (**left**) and $\pi/4$ (**right**). The parameter w of atomic motion is ignored and value of $\chi = 0.3, 1, 3$ (Non-linear Kerr).

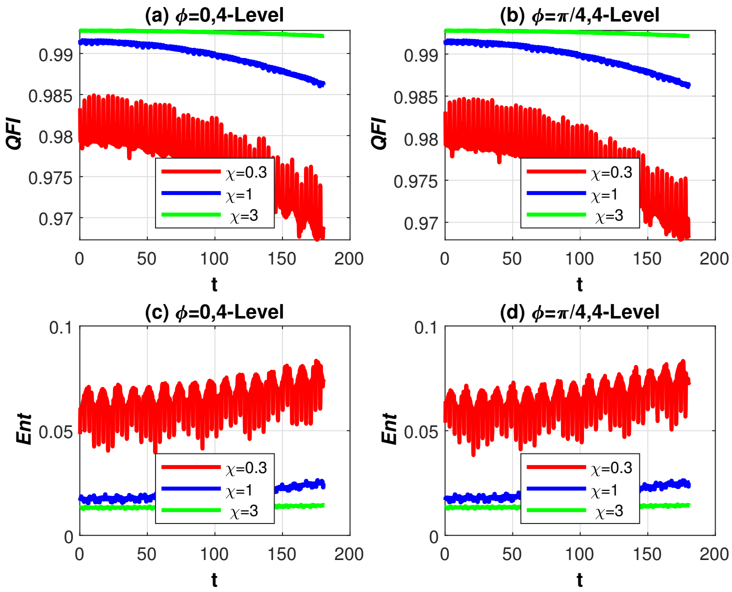

Figure 4. (Color online) The QFI (**top**) and von Neumann entropy (**bottom**) as a function of time for a system of four-level atom having interaction with coherent field for $|\alpha|^2 = 6$ and the phase shift estimator parameter $\phi = 0$ (**left**) and $\pi/4$ (**right**). The parameter w of atomic motion is ignored and value of $\chi = 0.3, 1, 3$ (Non-linear Kerr).

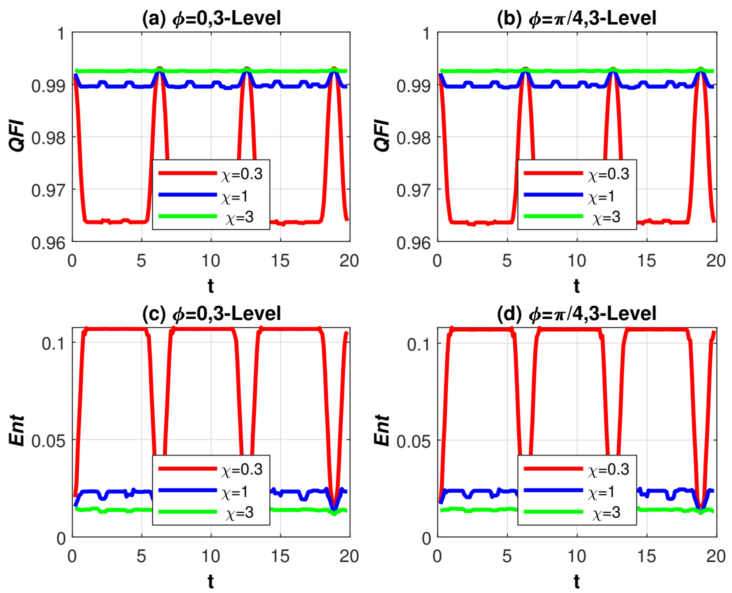

Figure 5. (Color online) The QFI (**top**) and von Neumann entropy (**bottom**) as a function of time for a system of three-level atom having interaction with coherent field for $|\alpha|^2 = 6$ and the phase shift estimator parameter $\phi = 0$ (**left**) and $\pi/4$ (**right**). The parameter $w = 1$ and value of $\chi = 0.3, 1, 3$ (Non-linear Kerr).

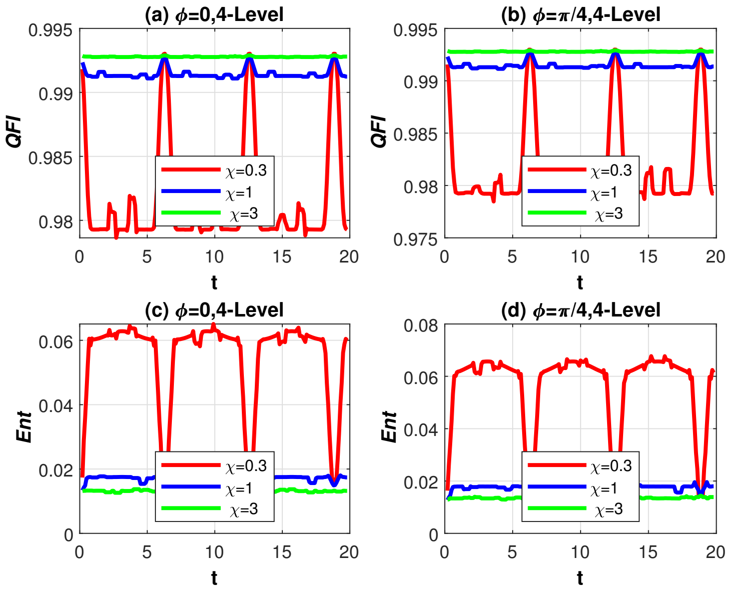

Figure 6. (Color online) The QFI (**top**) and von Neumann entropy (**bottom**) as a function of time for a system of four-level atom having interaction with coherent field for $|\alpha|^2 = 6$ and the phase shift estimator parameter $\phi = 0$ (**left**) and $\pi/4$ (**right**). The parameter w of atomic motion is 1 and value of $\chi = 0.3, 1, 3$ (Non-linear Kerr).

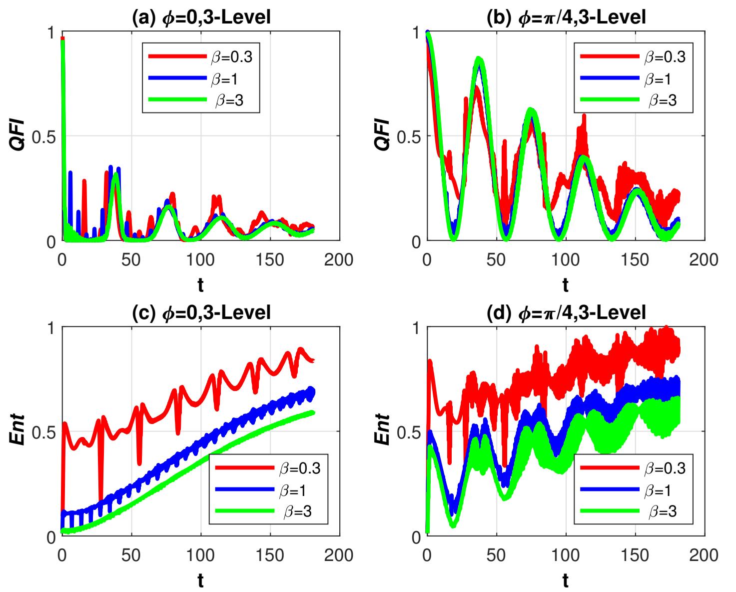

Figure 7. (Color online) The QFI (**top**) and von Neumann entropy (**bottom**) as a function of time for a system of four-level atom having interaction with coherent field for $|\alpha|^2 = 6$ and the phase shift estimator parameter $\phi = 0$ (**left**) and $\pi/4$ (**right**). The parameter w of atomic motion is ignored and value of $\beta = 0.3, 1, 3$ (Stark effect).

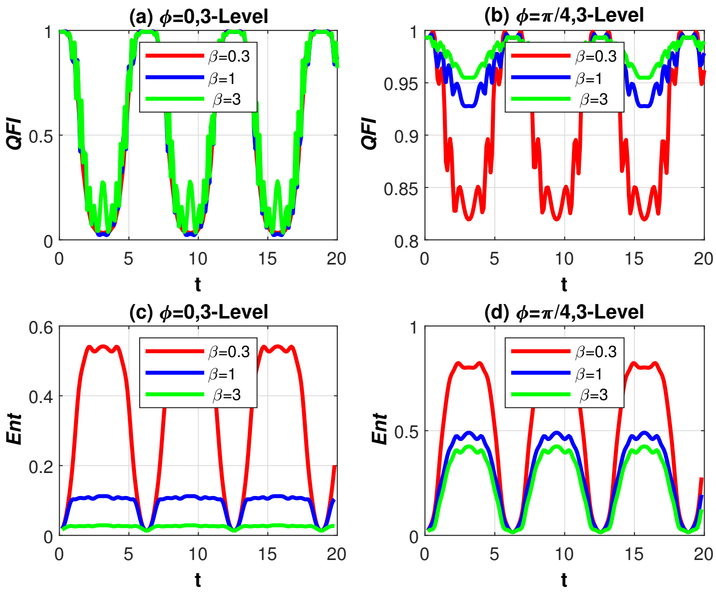

Figure 8. (Color online) The QFI (**top**) and von Neumann entropy (**bottom**) as a function of time for a system of three-level atom having interaction with coherent field for $|\alpha|^2 = 6$ and the phase shift estimator parameter $\phi = 0$ (**left**) and $\pi/4$ (**right**). The parameter w of atomic motion is 1 and value of $\beta = 0.3, 1, 3$ (Stark effect).

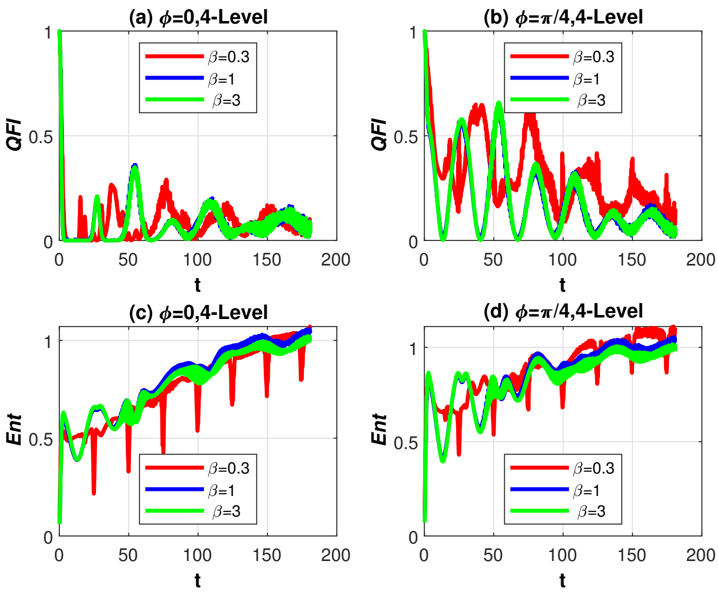

Figure 9. (Color online) The QFI (**top**) and von Neumann entropy (**bottom**) as a function of time for a system of four-level atom having interaction with coherent field for $|\alpha|^2 = 6$ and the phase shift estimator parameter $\phi = 0$ (**left**) and $\pi/4$ (**right**). The parameter w of atomic motion is ignored and value of $\beta = 0.3, 1, 3$ (Stark effect).

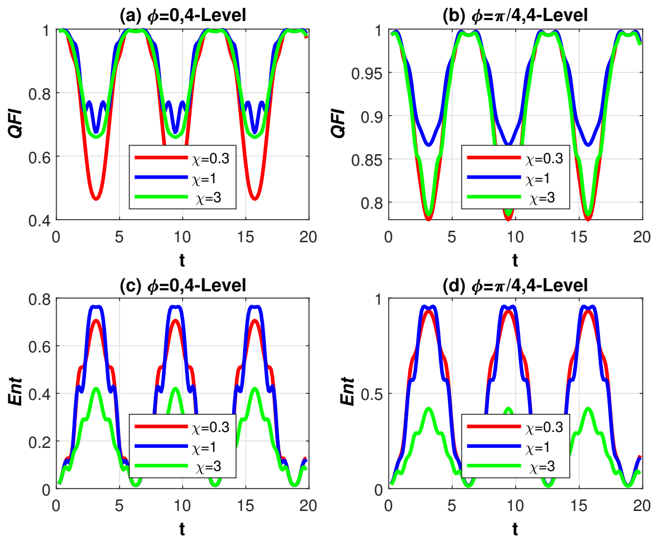

Figure 10. (Color online) The QFI (**top**) and von Neumann entropy (**bottom**) as a function of time for a system of four-level atom having interaction with coherent field for $|\alpha|^2 = 6$ and the phase shift estimator parameter $\phi = 0$ (**left**) and $\pi/4$ (**right**). The parameter w of atomic motion is 1 and value of $\beta = 0.3, 1, 3$ (Stark effect).

5. Conclusions

We studied the dynamical evolution of quantum Fisher information. We investigated numerically the dynamics of quantum Fisher information (QFI) and entanglement for three- and four-level atomic systems interacting with a coherent field under the effect of Stark shift and Kerr medium. It was observed that the Stark shift and Kerr-like medium play a prominent role during the time evolution of the quantum systems. The non-linear Kerr medium has a stronger effect on the dynamics of QFI as compared to the quantum entanglement (QE). QFI is heavily suppressed by increasing the value of Kerr parameter. This behavior was found comparable in the case of three- and four-level atomic system coupled with a non-linear Kerr medium. However, QFI and QE maintain their periodic nature under the effect of atomic motion. On the other hand, the local maximum value of QFI and von Neumann entropy (VNE) decrease gradually under the Stark effect. Moreover, no prominent difference in the behavior of QFI and QE was observed for three- and four-level atoms while increasing the value of Stark parameter. However, three- and four-level atomic systems were found equally prone to the non-linear Kerr medium and Stark effect. Moreover, three- and four-level atomic systems were found fully prone to the Kerr-like medium and Stark effect.

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